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An Analytical Study of the Effect of Viscosity on the Stability of an Infinite Fluid Slab Carrying a Uniform External Current

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1. INTRODUCTION

A plane infinite slab of thickness $2d$ of weakly conducting, nonmagnetic, incompressible fluid of density ρ, surface tension T, and kinematic viscosity ν, has one of its surfaces free and the other in contact with a plane insulating rigid boundary. A uniform external gravitational force of intensity g acts on the fluid, in a direction perpendicular to the rigid boundary from the free surface. An external current of uniform volume density is passed parallel to its surfaces and is closed symmetrically at infinity. The total current passing through the system at any instant of time is maintained constant by means of the external source. All the quantities are measured in rationalized M.K.S. units.

We shall consider the stability of the static equilibrium of the system just described by normal mode analysis. This is an extension to the viscous case of the study of the stability of a similar but inviscid system by Murty [1].

The dispersion relations for viscous systems are often transcendental in the frequency ω. While dealing with a transcendental dispersion relation of the type $D(\omega, k) = 0$, it is customary, at first, to find the so-called 'neutral' wave numbers defined by the roots of the equation $D(0, k) = 0$, and to assume that the system is alternately stable and unstable for the ranges of the wave number variable k separated by these so-called neutral wave numbers. The following two assumptions are implied in such a treatment: (i) that each of the so-called 'neutral' wave numbers is a "transitional" wave number (i.e., either the value 'zero' of ω is unique; or if there are other values of ω, they all have their real parts negative); and that (ii) there are no transtional wave numbers of the other type, namely, those in which the transition from stability to instability takes place through a state of oscillations with constant amplitude.

Since the transcendental equations may have infinite number of roots in the plane of the complex frequency ω, it is difficult to substantiate such assumptions by mahtematical analysis. There may be cases where one or both of these assumptions may really fail. The failure of the first kind of assumption in the discussion of a transcendental dispersion relation by Dungey and Loughhead [2] was reported by Tayler [3].

The roots of the dispersion relation as an equation in ω for a given value of k will be called as 'eigenfrequencies' corresponding to that k. We shall see later that in the linear approximation, the actual behavior of a system for a given k is ultimately determined by the "dominant eigenfrequency"; namely, the one whose real part is greater than or at least equal to that of every other eigenfrequency corresponding to the same k. Thus, one need not determine each and every root of the dispersion relation; but it is necessary to obtain so much information about their number and distribution in the ω-plane, as will enable one to identify the "dominant eigenfrequency." This, we shall attempt to achieve in this paper while studying the stability of the system described earlier. Such probes were carried out by Bellman and Pennington [4] and by Tayler [5], respectively, in cases of nontranscendental and a transcendental dispersion relation, which they had come across in their works.

2. The Basic Field Equations and the Boundary Conditions

For the reference of the various fields, we choose a right-handed rectangular cartesian coordinate system with origin midway between the surfaces of the slab, with positive y-axis in the direction of the current, and the positive z-axis along the normal to the free surface pointing from the rigid surface to the free surface. The equations for the rigid and the free surfaces are $z = -d$ and $z = +d$, respectively. The symbols 1_x, 1_y, 1_z will be used to represent unit vectors along the respective axes.

Referred to above system, the field quantities have to satisfy the following field equations and boundary conditions.

Field equations

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\nabla})\,\mathbf{V}\right] = -\boldsymbol{\nabla}P + \mathbf{J}\times\mathbf{B} - \rho g 1_z + \nu\rho\nabla^2\mathbf{V}, \tag{2.1}$$

$$\boldsymbol{\nabla}\cdot\mathbf{V} = 0, \tag{2.2}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0, \tag{2.3}$$

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J}, \tag{2.4}$$

$$\boldsymbol{\nabla}\cdot\mathbf{J} = 0, \tag{2.5}$$

and

$$\boldsymbol{\nabla}\times\mathbf{J} = 0, \tag{2.6}$$

where V, P, $\mathbf{J}$, $\mathbf{B}$ are respectively the fields of velocity, pressure, current density, and the magnetic field, respectively; μ_0 is the permeability of space (all are in rationalised M.K.S. system).

Equation (2.6) is the form taken by the induction equation in the case of weakly conducting liquids. It may be noted that this approximation is good even for fluids like mercury or liquid sodium in laboratory scales where the characteristic dimension of the system is of the order of a few millimeters and the characteristic time is of the order of a few milliseconds.

Boundary Conditions

The various fields are subject to the boundary condtions as following. For current density:

At both the free and the rigid boundaries

$$\mathbf{J} \cdot \hat{n} = 0,$$

where $\hat{n}$ is a unit vector normal to the boundary. As stated in the introduction, the current density is also subject to the constraint that the total strength of the current is maintained constant externally.

For Magnetic Fields

The magnetic fields must be continuous across both the boundary surfaces, and the fields outside the fluid must vanish at infinity. For velocity and pressure fields:

At the rigid boundary:

$$\mathbf{V} = 0.$$

At the free boundary:

$$\text{(i)} \qquad \frac{DF}{Dt} = 0, \tag{2.7}$$

$$\text{(ii)} \qquad \boldsymbol{S} \cdot \mathbf{n} = -P_T \hat{n},$$

where $F(x, y, z, t) = 0$ is the equation of the free boundary, D/Dt is the time derivative following the fluid motion, $\boldsymbol{S}$ is the viscous stress tensor, $\hat{n}$ is the unit normal vector to the surface, and P_T is the excess pressure on the liquid side of the boundary due to surface tension and the curvature. Equation (2.7) means that the velocity of evolution of any element of the surface normal to itself is same as that of the fluid element adjacent to it in that direction.

3. The Equilibrium State

The equilibrium fields are given by

$$\mathbf{V}_e = 0, \tag{3.1}$$

$$\mathbf{J}_e = J\mathbf{1}_y, \tag{3.2}$$

$$\mathbf{B}_e = B\mathbf{1}_x, \tag{3.3}$$

$$B = \mu_0 Jz, \qquad \text{for} \qquad -d \leqslant z \leqslant d, \tag{3.3a}$$

$$= \mu_0 Jd, \qquad \text{for} \qquad z > d, \tag{3.3b}$$

$$= -\mu_0 Jd, \qquad \text{for} \qquad z < -d, \tag{3.3c}$$

and

$$P_e = \frac{\mu_0 J^2(d^2 - z^2)}{2} + \rho g(d - z). \tag{3.4}$$

4. The Perturbation

In order to study the stability of the equilibrium, we suppose that the free surface is perturbed in such a way that its shape at an instant 't' of time is given by

$$z = d + a \exp(\omega t + i\phi),$$

where $\phi = k_x x + k_y y$ and $a \ll d$. We assume that all the field quantities after perturbation take the form

$$Q = Q_e + q\left(\frac{a}{d}\right) + \cdots,$$

where the space-time dependence of q is given by

$$q = q(z) \exp(\omega t + i\phi).$$

All the quantities will be considered only up to first order in (a/d).

5. The Solutions for the Perturbed Current Density and Magnetic Fields

From (2.5) and (2.6) together with the boundary condition and the constraint for the electric current density, the perturbed current density, to the first order in (a/d), can be shown to be given by

$$\mathbf{J} = \mathbf{J}_e + \frac{a}{d}\,\mathbf{j}(z) \exp(\omega t + i\phi), \tag{5.1}$$

where

$$j_x(z) = k_x A_{1,2} {\cosh \atop \sinh} kz,$$

$$j_y(z) = k_y A_{1,2} {\cosh \atop \sinh} kz,$$

and

$$j_z(z) = -ikA_{1,2} {\sinh \atop \cosh} kz.$$

$$A_{1,2} {\cosh \atop \sinh} kz \qquad \text{denotes} \qquad A_1 \cosh (kz) + A_2 \sinh (kz) \cdots \text{etc.}$$

(This notation will be freely used throughout hereafter.) A_1 and A_2 are constants given by

$$A_1 = -\frac{k_y J}{2k \sinh (kd)}, \qquad \text{and} \qquad A_2 = -\frac{k_y J}{2k \cosh (kd)},$$

where

$$k = +\sqrt{(k_x^2 + k_y^2)}.$$

Equations (2.3) and (2.4) together with the boundary conditions determine the perturbed magnetic field which, to the first order, is given by

$$\mathbf{B} = \mathbf{B}_e + \frac{a}{d}\, \mathbf{b}(z) \exp (\omega t + i\phi),$$

where

$$b_x(z) = \frac{k_x}{k} C_{1,2} {\sinh \atop \cosh} (kz) + \frac{\mu_0 k_y}{k} A_{1,2} {\sinh \atop \cosh} (kz),$$

$$b_y(z) = \frac{k_y}{k} C_{1,2} {\sinh \atop \cosh} (kz) - \frac{\mu_0 k_x}{k} A_{1,2} {\sinh \atop \cosh} (kz),$$

$$b_z(z) = -iC_{1,2} {\cosh \atop \sinh} (kz),$$

and

$$C_1 = C_2 = -\frac{k_x}{2k} \mu_0 J \exp (-kd).$$

6. The Perturbed Pressure and Velocity Fields

Taking the divergence of (2.1) we have

$$\nabla^2 P = -\mu_0 \mathbf{J}^2,$$

the solution of which in the form

$$P = P_e + \frac{a}{d}\, p(z) \exp (\omega t + i\phi) \tag{6.1}$$

to the first order in (a/d) can readily be found to give

$$p(z) = D_{1,2} \frac{\cosh}{\sinh} (kz) - \frac{k_y}{k} \mu_0 J z A_{1,2} \frac{\cosh}{\sinh} (kz),$$

where D_1 and D_2 are constants to be determined along with other constants in velocity field. By substituting (6.1) in the x-components of Eq. (2.1), a solution of the form

$$V_x = \frac{a}{d} \alpha(z) \exp (\omega t + i\phi),$$

to the first order in (a/d), can be obtained, where $\alpha(z)$ is given by the differential equation

$$\left(\frac{d^2}{dz^2} - \lambda^2\right) \alpha(z) = \frac{i}{\nu\rho} \left(k_x D_{1,2} + JC_{1,2} - \frac{k_x k_y}{k} \mu_0 z A_{1,2}\right) \frac{\cosh}{\sinh}(kz), \qquad (6.2)$$

where

$$\lambda^2 = k^2 + \frac{\omega}{\nu}.$$

The general solution of (6.4) is

$$\alpha(z) = \alpha_{1,2} \frac{\cosh}{\sinh} \lambda z + \frac{i}{\omega\rho} \Big[(- k_x D_{1,2} - JC_{1,2}) + \frac{k_x k_y}{k} \mu_0 J \left(z A_{1,2} + \frac{2\nu k}{\omega} A_{1,2}\right)\Big] \frac{\cosh}{\sinh} (kz), \qquad (6.3)$$

where α_1, α_2 are coefficients to be determined along with other coefficients, and

$$\lambda = + \sqrt{k^2 + \frac{\omega}{\nu}},$$

where, for the sake of definiteness, the square root with positive real part is chosen.

The inhomogeneous part of the solution (6.3) is proper only when

(i) $\omega \neq 0$, and

(ii) $\omega \neq - \nu k^2$.

The implications of $\omega = 0$ and $\omega = - \nu k^2$ are considered in Appendix A.

The solution for the y-component of velocity is similar, and involves two more coefficients β_1, β_2 to be determined along with α_1, α_2 and D_1, D_2.

The solution for the z-component of velocity follows immediately from the continuity equation, and does not involve any new undetermined coefficients.

7. The Dispersion Relation

The pressure and velocity fields determined in last article are subject to the boundary conditions in Section 2. To the first order in (a/d), these yield seven inhomogeneous linear equations in the six coefficients D_1, D_2, α_1, α_2, β_1, and β_2, hitherto undetermined. The following dispersion relation is then obtained as the condition of consistency of these seven equations in six unknowns. On simplification, the said relation (dispersion relation) takes the form

$$\begin{aligned}\mathfrak{D}(\omega, k) \equiv \omega^2(\lambda C_{2\lambda d}C_{2kd} - kS_{2\lambda d}S_{2kd}) + 4\nu k^2\omega[\lambda(C_{2\lambda d}C_{2kd} - 1) - kS_{2\lambda d}S_{2kd}\\ + 4\nu^2k^3[2\lambda k(C_{2\lambda d}C_{2kd} - 1)\\ + (\lambda^2 + k^2)\, S_{2\lambda d}S_{2kd}]\\ + Z(\lambda C_{2\lambda d}S_{2kd} - kS_{2\lambda d}C_{2kd}) = 0,\end{aligned} \tag{7.1}$$

where

$$Z = \left[\frac{Tk^3}{\rho} + gk + \frac{k_x{}^2\mu_0J^2}{2k^2\rho}(2kd - 1)\right], \tag{7.2}$$

$$C_{2\lambda d} = \cosh(2\lambda d), \qquad S_{2\lambda d} = \sinh(2\lambda d),$$
$$C_{2kd} = \cosh(2kd), \qquad S_{2kd} = \sinh(2kd).$$

The expression on the right-hand side of Eq. (7.2) is same as that occurring in the square bracket in dispersion relation for the inviscid case obtained by Murty [1] with reads, in our notation, as

$$\omega^2 = -\left[\frac{Tk^3}{\rho} + gk + \frac{k_x{}^2\mu_0J^2}{2k^2\rho}(2kd - 1)\right]\frac{S_{2kd}}{C_{2kd}}. \tag{7.3}$$

Returning to Eq. (7.1) we choose the dimensionless variables

$$z = 2\lambda d \qquad \text{and} \qquad x = 2kd;$$

and write the relation (7.1) in terms of these. We obtain then

$$\begin{aligned}\omega^2d(zC_zC_x - xS_zS_x) + 4\nu k^2\omega d[z(C_zC_x - 1) - xS_zS_x]\\ + 2\nu^2k^3[2zx(C_zC_x - 1) - (z^2 + x^2)\, S_zS_x]\\ + Zd(zC_zS_x - xS_zC_x) = 0.\end{aligned}$$

This can be further simplified by expressing it in terms of z/x. We choose

$$\zeta = \frac{z}{x} = \frac{\lambda}{k} = \sqrt{1 + \left(\frac{\omega}{\nu k^2}\right)},$$

that is,

$$\omega = \nu k^2(\zeta^2 - 1). \tag{7.4}$$

The dispersion relation then becomes

$$(\zeta^5 + 2\zeta^3 + 5\zeta)\, C_{\zeta x} C_x - (\zeta^4 + 6\zeta^2 + 1)\, S_{\zeta x} S_x - 4(\zeta^3 + \zeta) + X(\zeta C_{\zeta x} S_x - x S_{\zeta x} C_x) = 0, \tag{7.5}$$

where

$$X = \frac{8d^3}{\nu^2 x^4}\left[\frac{T}{4\rho d^2}\, x^3 + \left(g + \frac{\mu_0 J^2 d}{\rho}\sin^2\theta\right) - \frac{\mu_0 J^2 d}{\rho}\sin^2\theta\right], \tag{7.6}$$

θ being the inclination of the wave vector to the direction of the equilibrium current.

We note that the following quantities appear in the dispersion relation *only through the function* $X(x)$:

(i) the properties of the liquid other than viscosity, viz. ρ and T;
(ii) other parameters of the system, viz. d, g, J; and
(iii) the direction of the wave vector given by θ.

Since all directions of the wave vector can be considered by varying θ from 0 to 2π, it is sufficient to consider only positive values of x.

To study the variation of ω with respect to x, we first consider all the above quantities to have some fixed value, so that the dispersion relation (7.5) can briefly be written as

$$\Phi(\zeta, x) = 0,$$

where

$$\Phi(\zeta, x) = (\zeta^5 + 2\zeta^3 + 5\zeta)\, C_{\zeta x} C_x - (\zeta^4 + 6\zeta^2 + 1)\, S_{\zeta x} S_x - 4(\zeta^3 + \zeta) + X(\zeta C_{\zeta x} S_x - x S_{\zeta x} C_x).$$

When referred to as a function of ζ for a fixed x, it will be often written as $\Phi(\zeta)$.

8. The Dominant Root of $\Phi(\zeta, x) = 0$ and the Dominant Eigenfrequency

The function $\Phi(\zeta, x)$ being transcendental in ζ, there may, in general be, for a given x, several values of ζ satisfying $\Phi(\zeta, x) = 0$. All these values of ζ need not correspond to physical values of ω (see Appendix A and Appendix

B). It may happen, however, that more than one of them gives according to (7.4) corresponding physical values of ω by the rule

$$\omega_r = \nu k^2[(\xi_r^2 - \eta_r^2 - 1) + 2i\xi_r\eta_r],$$

where $\zeta_r = \xi_r + i\eta_r$ ($r = 1, 2, \ldots$) are different roots.

The actual space-time-dependence of all the perturbations will then be of the type

$$\sum_r \mathscr{A}_r(k, z) \exp(\omega t + i\phi). \tag{8.1}$$

After sufficient lapse of time, the term with maximum real part of ω_r will be dominant. Till this happens, the nonlinear effects can be assumed not to enter into the picture, since the amplitude of perturbation is arbitrarily small. In terms of ζ_r, this means that the actual behavior of the system, for a given x, is ultimately determined by that root ζ_r, for which $(\xi_r^2 - \eta_r^2)$ is maximum. We call such a root as ***dominant root*** and the corresponding value of ω_r as ***dominant eigenfrequency***.

In order to be able to determine the dominant root of $\Phi(\zeta, x) = 0$ for each given value of x, we investigate the distribution of *all* the roots of $\Phi(\zeta, x) = 0$ in the complex ζ-plane. For this, we first search, in the following two sections, for as many real and pure-imaginary roots of $\Phi(\zeta) = 0$ as possible. In doing so, and in the subsequent investigation, we shall be aided by the following symmetry properties of $\Phi(\zeta)$, and its real and imaginary parts $R(\xi, \eta)$ and $I(\xi, \eta)$, respectively:

$$\left.\begin{aligned} \Phi(-\zeta, x) &= -\Phi(\zeta, x), \\ R(-\xi, \eta, x) &= -R(\xi, \eta, x), \\ R(\xi, -\eta, x) &= +R(\xi, \eta, x), \\ I(-\xi, \eta, x) &= +I(\xi, \eta, x), \\ \text{and} \qquad I(\xi, -\eta, x) &= -I(\xi, \eta, x), \end{aligned}\right\} \text{for every value of } x.$$

As a consequence of these, the real and pure imaginary roots of $\Phi(\zeta) = 0$ occur, for each x, in pairs of the type $\{\pm\, \xi_r\}$ and $\{\pm\, i\eta_r\}$, respectively, while the roots which are neither purely real nor purely imaginary occur, for each x, in quadrets of the type $\{\xi_r + i\eta_r, \xi_r - i\eta_r - \xi_r + i\eta_r, -\xi_r - i\eta_r\}$. The roots in the same pair give one and the same value of ω, while the roots in the same quadraet give two values of ω, which are complex conjugate of each other.

9. The Real Roots of $\Phi(\zeta) = 0$

As mentioned in the preceding section, we shall search, in this section, for as many real roots of $\Phi(\zeta, x) = 0$, for any given x, as possible.

First of all, we see that -1, 0 and $+1$ are the roots of $\Phi(\zeta, x) = 0$ for every value of x. As we shall see later, these roots have no physical significance so long as they are of first-order. Nonetheless, they do form, for every x, a subset of the set of all the roots of $\Phi(\zeta) = 0$.

To find if there are any real roots of $\Phi(\zeta) = 0$ in the intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$, we consider the signs of $\Phi(-\infty, x)$, $\Phi(-1-\delta, x)$, $\Phi(-1+\delta, x)$, $\Phi(-\delta, x)$, $\Phi(\delta, x)$, $\Phi(1-\delta, x)$, $\Phi(1+\delta, x)$, and $\Phi(\infty, x)$ for a small positive δ. This can be done since $\Phi(\zeta)$ is real when ζ is real. In view of the symmetry properties mentioned in the last section, it is sufficient to consider only the signs of $\Phi(\delta, x)$, $\Phi(1-\delta, x)$, $\Phi(1+\delta, x)$, and $\Phi(\infty, x)$. The expressions for these, in terms of x, are as following:

$$\Phi(\delta, x) = \delta\Delta_{01}(x) + \delta^3\Delta_{03}(x) + \cdots, \tag{9.1a}$$

$$\Phi(1-\delta, x) = -\delta\Delta_{11}(x) + \delta^2\Delta_{12}(x) + \cdots, \tag{9.1b}$$

$$\Phi(1+\delta, x) = \delta\Delta_{11}(x) + \delta^2\Delta_{12}(x) + \cdots, \tag{9.1c}$$

$$\Phi(\infty, x) = +\infty, \tag{9.1d}$$

where

$$\Delta_{01}(x) = (5C_x - xS_x - 4) - X(xC_x - S_x), \tag{9.2}$$

$$\Delta_{03}(x) = \left(2 + \frac{5x^2}{2}\right)C_x - \left(6x + \frac{x^3}{6}\right)S_x - X\left(\frac{x^2S_x}{2} - \frac{x^3C_x}{6}\right), \tag{9.3}$$

$$\Delta_{11}(x) = X(S_xC_x - x), \tag{9.4}$$

and

$$\Delta_{12}(x) = 4C_x^2 + 4S_x^2 + X(xS_x^2). \tag{9.5}$$

To determine the signs of $\Phi(\delta, x)$, $\Phi(1-\delta, x)$, $\Phi(1+\delta, x)$ we must study how the signs of $\Delta_{01}(x)$ and $\Delta_{11}(x)$ vary with respect to x, and what are the signs of $\Delta_{03}(x)$ and $\Delta_{12}(x)$, respectively, where $\Delta_{01}(x)$ and $\Delta_{11}(x)$ vanish. For this study of signs, we need consider the following properties of the functions X, $S_xC_x - x$, $5C_x - xS_x - 4$ and $xC_x - S_x$.

For X

From (7.6), it can be seen, with Descarte's rule of signs, that $X = 0$ has a *unique real positive value* x_c such that

$$X(x) < 0 \quad \text{for all} \quad x < x_c, \qquad X = 0 \quad \text{at} \quad x = x_c,$$

and

$$X(x) > 0 \qquad \text{for all} \qquad x > x_c \,. \tag{9.6}$$

Further, since $X(x) < 0$ at $x = 0$ and $X > 0$ at $x = 1$, we conclude

$$x_c \leqslant 1 \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \tag{9.7}$$

For $S_x C_x - X$

From the series expansion of sinh $(2x)$, we see that

$$S_{2x} > 2x \qquad \text{for all} \qquad x > 0.$$

This means that

$$S_x C_x - x > 0 \qquad \text{for all} \qquad x > 0. \tag{9.8}$$

For $5C_x - xSx - 4$

The first five derivatives of this function with respect to x are: $4S_x - xC_x$, $3C_x - xS_x$, $2S_x - xC_x$, $C_x - xS_x$, and $-xC_x$, respectively. The last function does not vanish anywhere in the open interval $(0, \infty)$. Hence $C_x - xS_x$ can have *at the most* one positive zero. Since its signs as $x \to 0$ and $x \to \infty$ are opposite, we conclude that it has *exactly on positive zero*. The function $2S_x - xC_x$ therefore has *at the most* two positive zeros. But since its signs as $x \to 0$ and $x \to \infty$ are opposite, we conclude that the function $2S_x - xC_x$ has exactly one positive zero. Continuing in this way, it can be finally shown that the function $5C_x - xS_x - 4$ has a unique positive zero, say m. We then have

$$(5C_x - xS_x - 4) \gtreqless 0 \qquad \text{for} \qquad x \lesseqgtr m. \tag{9.9}$$

Since $(5C_x - xS_x - 4) > 0$ at $x = 1$ and < 0 as $x \to \infty$, we conclude that $m > 1$; and therefore from (9.7)

$$x_c < m. \tag{9.10}$$

For $xC_x - S_x$

From power series for this function, it can be immediately seen that

$$xC_x - S_x > 0 \qquad \text{for all} \qquad x > 0. \tag{9.11}$$

We are now in a position to consider the behavior of the signs of $\Delta_{01}(x)$ and $\Delta_{11}(x)$. From (9.4), (9.6), and (9.8) we see that

$$\begin{gathered} \Delta_{11}(x) < 0 \qquad \text{for all} \qquad x < x_c \,, \\ \Delta_{11}(x_c) = 0 \quad \text{and} \quad \Delta_{11}(x) > 0 \qquad \text{for all} \qquad x > x_c \,. \end{gathered} \tag{9.12}$$

We also find immediately that

$$\Delta_{12}(x_c) > 0. \tag{9.13}$$

From (9.2) and (9.9)-(9.11), we can deduce that

$$\Delta_{01}(x) > 0 \qquad \text{for all} \qquad x \leqslant x_c . \tag{9.14}$$

Further since $\Delta_{01}(x) < 0$ as $x \to \infty$, we conclude that the equation $\Delta_{01}(x) = 0$ *has at least one real root in* (x_c, ∞). It is difficult to examine analytically the exact number of the roots of $\Delta_{01}(x)$ in this range, since it depends in a complicated way on the relative values of the parameters T, g, ν, J etc. of the system. However, we can analyze the situation for all wave numbers upto the smallest positive real root of $\Delta_{01}(x) = 0$, and we shall be satisfied with it.

We shall define x_* as the positive real root of $\Delta_{01}(x) = 0$ if there is only one real root, and as the smallest positive real root, if there are more than one of them. We then have

$$\Delta_{01}(x) > 0 \qquad \text{for all} \qquad x < x_* , \tag{9.15a}$$

$$\Delta_{01}(x_*) = 0, \tag{9.15b}$$

and for $x > x_*$, and sufficiently near to it,

$$\Delta_{01}(x) < 0 \cdots . \tag{9.15c}$$

In view of (9.14) we have

$$x_c < x_* . \tag{9.16}$$

The sign of $\Delta_{03}(x)$ at x_* and other zeros of $\Delta_{01}(x)$ cannot be determined analytically, but we shall see that the stability criterion at these points is independent of that sign. From the behavior of the signs of $\Delta_{01}(x)$ and $\Delta_{11}(x)$ for varying values of x and from the signs of $\Delta_{03}(x)$ and $\Delta_{12}(x)$ where $\Delta_{01}(x)$ and $\Delta_{11}(x)$ respectively vanish, the sequence of the signs of $\Phi(\delta, x)$, $\Phi(1 - \delta, x)$, $\Phi(1 + \delta, x)$, and $\Phi(\infty, x)$ can be written down for the four cases: (i) $x < x_c$, (ii) $x = x_c$, (iii) $x_c < x < x_*$, (iv) $x = x_*$.

9.1. *For* $0 < x < x_c$

In the case when $x < x_c$, we find using Eqs. (9.1a)-(9.5), (9.12), (9.15a), and (9.16)

$$\Phi(0, x) = 0, \qquad \Phi(\delta) > 0, \qquad \Phi(1 - \delta) > 0, \qquad \Phi(1) = 0,$$
$$\Phi(1 + \delta) < 0 \qquad \text{and} \qquad \Phi(\infty) > 0.$$

This sequence of signs proves the presence of at least one real root, say, $\zeta_0 > 1$ (and by symmetry, one negative real root $-\zeta_0 < -1$) for $0 < x < x_c$

Hence at least the five real roots $-\zeta_0$, $-1, 0, +1, +\zeta_0$, exist. If any more positive real roots are present, they must occur in pairs on either side of 1; and by symmetry there must be similar pair or pairs on the negative ζ-axis. Hence the actual number of real roots of $\Phi(\zeta) = 0$ may be $5 + 4n$ where n is an integer $\geqslant 0$.

9.2. *At* $x = x_c$

When $x = x_c$ we obtain from Eqs. (9.1a)-(9.5), (9.12), (9.13), (9.15a), and (9.16)

$$\Phi(\delta) > 0 \qquad \text{and is of the order of } \delta,$$

$$\Phi(1 - \delta) > 0 \qquad \text{and is of the order of } \delta^2,$$

$$\Phi(1) = 0,$$

$$\Phi(1 + \delta) > 0 \qquad \text{and is of the order of } \delta^2,$$

and

$$\Phi(\infty) > 0.$$

This sequence of signs shows the presence of the following five roots: (-1 twice, 0, $+1$ twice). From an argument like the one in 9.1 we conclude that the actual number of real roots is $5 + 4n$ where n may be any integer $\geqslant 0$.

9.3. *For* $x_c < x < x_*$

In this case we have

$$\Phi(\delta) > 0 \qquad \text{and is of the order of } \delta,$$

$$\Phi(1 - \delta) < 0 \qquad \text{and is of the order of } \delta,$$

$$\Phi(1) = 0,$$

$$\Phi(1 + \delta) > 0 \qquad \text{and is of the order of } \delta,$$

and

$$\Phi(\infty) > 0.$$

This sequence of signs shows the presence of at least one root, say ζ_0, between 0 and $+1$, and by symmetry, at least one root $-\zeta_0$ between -1 and 0. Hence there exist at least the following five real roots: (-1, $-\zeta_0$, 0, $+\zeta_0$, $+1$). Here also the total number of real roots is $5 + 4n$ where n may be any integer $\geqslant 0$.

9.4. *At* $x = x_*$

If $x = x_*$, we find from considerations of the signs of $\Delta_{11}(x_*)$ etc. that

$$\begin{aligned} \Phi(0) &= 0, && \Phi(\delta) \text{ is of the order of } \delta^3, \\ \Phi(1-\delta) &< 0 && \text{and is of the order of } \delta, \\ \Phi(1) &= 0, && \\ \Phi(1+\delta) &> 0 && \text{and is of the order of } \delta, \end{aligned}$$

and

$$\Phi(\infty) > 0.$$

This sequence of signs shows the presence of at least the following five real roots: (-1, 0 thrice, $+1$). The total number of real roots is thus again $5 + 4n$ where n is any integer $\geqslant 0$. It can be seen that the result of this subsection is true for all x when $\Delta_{01}(x) = 0$.

10. The Pure Imaginary Roots

We shall now search for the pure imaginary roots of $\Phi(\zeta, x) = 0$ in the ζ-plane. We have

$$\begin{aligned} \Phi(i\eta) = i[(\eta^5 - 2\eta^3 + 5\eta)\, C_x \cos(\eta x) - (\eta^4 - 6\eta^2 + 1)\, S_x \sin(\eta x) \\ - 4(\eta - \eta^3) \\ + X\{\eta S_x \cos(\eta x) - C_x \sin(\eta x)\}] \\ = iF(\eta) \text{ say.} \end{aligned}$$

At the points $\eta = 2K\pi/x$, ($K = \pm 1, \pm 2, \pm 3, \ldots$).

$$F(\eta) = F_+(\eta), \tag{10.1}$$

where

$$F_+(\eta) = \eta\{C_x \eta^4 - 2(C_x - 2)\,\eta^2 + (5C_x - 4 + XS_x)\} \tag{10.1a}$$

and at the points

$$\eta = \frac{(2K-1)\,\pi}{x}, \qquad (K = \pm 1, \pm 2, \pm 2, \ldots),$$

$$F(\eta) = -F_-(\eta), \tag{10.2}$$

where

$$F_-(\eta) = \eta\{C_x\eta^4 - 2(C_x + 2)\,\eta^2 + (5C_x + 4 + XS_x)\}. \qquad (10.2a)$$

The expressions in the brackets (10.1a) and (10.2a) are quadratic in η^2 and both these have the same discriminant, *viz.*

$$\Delta(x) = -\,4S_x(4S_x + XC_x).$$

Now, $X \geqslant 0$ for $x \geqslant x_c$. Therefore, $\Delta(x) < 0$ for all $x \geqslant x_c$. On the other hand, for sufficiently small positive x, $\Delta(x) > 0$. Therefore, there exists x_{**}, the greatest real root of $\Delta(x) = 0$ less than x_c, such that

$$\Delta(x) < 0 \qquad \text{for all} \qquad x > x_{**},$$

and

$$\Delta(x_{**}) = 0. \qquad (10.3)$$

Thus, for all $x \geqslant x_{**}$, neither $F_+(\eta)$ nor $F_-(\eta)$ has any nonzero real zeros, i.e.,

$$F_+(\eta), F_-(\eta) \gtrless 0 \qquad \text{for} \qquad \eta \gtrless 0. \qquad (10.4)$$

From (10.1), (10.2), and (10.4) we see that if $x \geqslant x_{**}$,

$$F(\eta) > 0 \qquad \text{for all} \qquad \eta = \frac{2K\pi}{x}, \qquad K = 1, 2, 3, \ldots$$

and

$$F(\eta) < 0 \qquad \text{for all} \qquad \eta = \frac{(2K-1)\,\pi}{x}, \qquad K = 1, 2, 3, \ldots.$$

Hence for all $x \geqslant x_{**}$, there exists at least one root of $F(\eta) = 0$ in each of the intervals $(\pi/x, 2\pi/x)$, $(2\pi/x, 3\pi/x)$, $\cdots$ etc. and at least one root in each of the intervals $(-\pi/x, -2\pi/x)$, $(-2\pi/x, -3\pi/x)$, $\cdots$ etc. We shall call this ***result I*** in future references.

We cannot say anything about the intervals $(0, \pi/x)$ and $(-\pi/x, 0)$ without considering the signs of $F(-\pi/x)$, $F(-\delta)$, $F(\delta)$, and $F(\pi/x)$. We see that $F(-\pi/x) > 0$, $F(\pi/x) < 0$ and $F(\pm\delta) = \pm\,\delta\Delta_{01}(x)$. *Hence if x also satisfies* $x < x_*$, so that $\Delta_{01}(x) > 0$, then $F(-\delta) < 0$ and $F(\delta) > 0$; and then at least one root will be present in each of the open intervals $(0, \pi/x)$, and $(-\pi/x, 0)$. We shall call this ***result II.***

11. Possible Number of Roots

Having detected a number of roots of $\Phi(\zeta, x) = 0$ for different x, it is important to see whether there can be any more roots. The following theorem due to L. S. Pontryagin, Theorem 3, Ref. [6], is of great value here.

Pontryagin's Theorem

Let $f(z, u, v)$ be a polynomial in the variables z, u, v with real coefficients, and let it be written in the form

$$f(z, u, v) = \sum_{m=0}^{r} \sum_{n=0}^{s} B_{mn} z^m \Phi_m^{(n)}(u, v), \tag{11.1}$$

where $\Phi_m^{(n)}(u, v)$ is a homogeneous polynomial of degree n in u and v. Let the principal term $B_{rs}\Phi_r^{(s)}(u, v)$ be present ($B_{rs} \neq 0$). Further let $\Phi_*{}^s(u, v)$ be the coefficient of z^r in (11.1), i.e., let

$$\phi_*^{(s)}(u, v) = \sum_{n=0}^{s} \Phi_r^{(n)}(u, v);$$

and let ϵ be such that $\Phi_*^{(s)}$ (cos z, sin z) does not vanish anywhere on the vertical line $\mathrm{Re} \cdot z = \epsilon$. Then: in the strip $-2K\pi + \epsilon \leqslant \mathrm{Re}\, z \leqslant 2K\pi + \epsilon$, the function $f(z, \cos z, \sin z)$ will have, starting from sufficiently large integral value of K, exactly $4sK + r$ zeros.

Applying this theorem to our function $\Phi(\zeta)$, we see that, starting from some sufficiently large integral value of K, the function $\Phi(\zeta)$ will have exactly $4K + 5$ zeros in the strip $-2K\pi/x \leqslant \eta \leqslant 2K\pi/x$ for every fixed value of x, where $\zeta = \xi + i\eta$.

12. Combination of the Results of Sections 9, 10, and 11

Comparing the roots of $\Phi(\zeta) = 0$ detected in Sections 9 and 11 with their distribution expected according to Section 11, we have the following information about their distribution for the cases

(i) $x_{**} \leqslant x < x_*$,
(ii) $x = x_*$, and
(iii) $x > x_*$.

12.1. *For* $x_{**} \leqslant x < x_*$

For values of x in this range 5 real roots are detected in Section 9. Moreover, both *results I and II* of Section 10 apply in this case, which together account for $4K$ pure imaginary roots in the strip $-2K\pi/x \leqslant \eta \leqslant 2K\pi/x$, for any integral K. Thus, in view of Section 11, all the roots of $\Phi(\zeta) = 0$ have been detected in this case; and hence the words 'at least' should in fact be replaced by the word '*exactly*' in Sections 9.1-4 and results I and II of Section 10.

Discarding then the roots ± 1 when $x \neq x_c$ and considering them when $x = x_c$, the reasons for which are given in Appendices A and B, we further see that

(i) $\pm \zeta_0$ of Section 9.1 are the dominant roots for $x_{**} \leqslant x < x_c$,
(ii) ± 1 are the dominant roots at $x = x_c$,
(iii) $\pm \zeta_0$ of Section 9.3 are the dominant roots for $x_c < x < x_*$.

12.2. *At* $x = x_*$

In this case $x > x_{**}$ is still true so that the *result I* of Section 10 still holds which accounts for $4K - 2$ roots in the strip $-2K\pi/x \leqslant \eta \leqslant 2K\pi/x$. There are also the roots ± 1 and a triple root at 0 (see 9.4). Thus $4K + 3$ roots are detected. Hence only two roots are missing.

The consideration of the sequences of signs $\Phi(\delta)$, $\Phi(1 - \delta)$, and $F(\delta)$, $F(\pi/x)$ shows that the remaining two roots are either both real (one each in the ranges $(-1, 0)$ and $(0, 1)$), or both imaginary (one each in the ranges $(-\pi/x, 0)$ and $(0, \pi/x)$ on η-axis) according as $\Delta_{03}(x_*)$ is positive or negative.

12.3. *For intervals of* x *beyond* x_* *in which* $\Delta_{01}(x) < 0$

Since $x > x_{**}$ is still true, this gives $4K - 2$ roots in $-2K\pi/x \leqslant \eta \leqslant 2K\pi/x$. The three real roots $0, \pm 1$ also exist. Thus in this case $4K + 1$ roots are detected. The remaining four roots, then, can (a) either be situated all on real axis with a pair on each side of 0, or (b) all on imaginary axis with a pair on each side of 0, or (*c*) symmetrically situated off the axes, one in each quadrant.

The distribution described here will be true for the whole interval (x_*, ∞) if x_* is the only positive root of $\Delta_{01}(x) = 0$. If there are more positive roots of $\Delta_{01}(x) = 0$, they are greater than x_* (by definition of x_*) and then the results in (12.1), (12.2) or (12.3) will apply to any interval according as $\Delta_{01}(x)$ is positive, zero, or negative in that interval.

13. The Physical Results

13.1. *The Instability of the System for* $x < x_c$

The presence, proved in (9.1) of the roots $\pm \zeta_0$ with $\zeta_0 > 1$ giving a real positive value $\nu k^2(\zeta^2 - 1)$ to ω, for all $x < x_c$ establishes the instability of the system for all $x < x_c$. If further $x_{**} \leqslant x < x_c$ it can be seen from the distribution of all the roots of the dispersion relation given in (12.1) that the real positive value $\nu k^2(\zeta^2 - 1)$ stated above is dominant in the sense of Section 8. Hence if $x_{**} \leqslant x < x_c$ the instability manifests as purely exponential growth of perturbations.

13.2. *The Stationary Point $x = x_c$ and its Uniqueness*

As will be shown in Appendix A, x_c is the only positive value of x, for which '0' is a possible value of ω. Though ordinarily the permanent roots ± 1 of $\Phi(\zeta) = 0$ are of no physical significance as also shown in Appendix A, we see, from (9.2), that at $x = x_c$ there is a 'double' root of $\Phi(\zeta) = 0$ at each of the points ± 1. That this must be interpreted as giving a value '0' of ω at $x = x_c$, is shown in Appendix B. Further from the distribution of all the roots of $\Phi(\zeta) = 0$, given in (12.1), ± 1 are obviously the dominant roots.

Thus, $x = x_c$ is a ***true stationary point*** and it is the only non-trivial wave number for which $\omega = 0$ is possible. Note that the term 'stationary' is used here in the sense of dominant frequency having both the real and imaginary parts zero.

13.3. *The Stability for $x < x \leqslant x_*$*

From the completely known distribution of roots of $\Phi(\zeta) = 0$ according to (12.1) for $x_c < x < x_*$ and from (12.2) at $x = x_*$, it follows that for x in this range the system is always stable. Since the dominant root ζ_0 gives real and negative ω, the stability manifests as purely exponential decay.

13.4. *Results for $x_* < x < \infty$*

No conclusion can be drawn on the basis of present analysis for those values of x at which $\Delta_{01}(x) < 0$. (Such values lie only beyond x_* by its definition.) This applies to the whole region (x_*, ∞) if x_* is the only finite positive real root of $\Delta_{01}(x) = 0$. If, however, there are more roots of this equation, we have (a) for intervals of x with $\Delta_{01}(x) > 0$, the conclusion of (13.3) is still valid according to (12.3); (b) for values of x at which $\Delta_{01}(x) = 0$, the conclusion of (13.3) is still valid according to (12.3); (c) for intervals of x with $\Delta_{01}(x) < 0$, no definite conclusion can be reached on the basis of present analysis.

13.5. *The Principle of Exchange of Stabilities*

We have proved in (13.1, 2, 3) that there is an actual transition from stability to instability through a stationary state at $x = x_c$ as x decreases through that value. Moreover, the instability sets in as a 'cellular convection' (see Chandrasekhar [7]), since all the eigenfrequencies are real for all x in (x_{**}, x_*) containing the point x_c. Thus the principle of exchange of stabilities holds for this transition.

14. THE LIMIT AS $\nu \to 0$

We shall now discuss the form of the dispersion relation as the viscosity tends to zero. We shall be interested in only those branches of ω as the multiple valued function of k, which, for each finite k, remains finite for finite ν and tend to nonzero limit as $\nu \to 0$. Hence we can take $(\nu k^2/\omega) \to 0$ as equivalent to $\nu \to 0$. Expressing $C_{\zeta x}$ and $S_{\zeta x}$ in terms of exp (ζx) and exp $(-\zeta x)$ and taking the limit, it can be verified that the limiting form of the dispersion relation as $\nu k^2/\omega \to 0$ is

$$\cosh x + \frac{Z}{\omega^2} \sinh x = 0$$

which is same as (7.3), the dispersion relation for a similar but inviscid system.

In the viscous case, all values of ω are real on both sides of the transition wave number x_c at least in the interval (x_{**}, x_*) surrounding it. In the inviscid case ω passes from a real value to a pure imaginary value through the value 0 as the wave number increases through the transitional wave number x_c. Then how can, the behavior of in the viscous case pass smoothly to that in the inviscid case as viscosity tends to zero? The answer to this question lies in the fact that for sufficiently small viscosity, x_* is the only root of $\Delta_{01}(x) = 0$ and it tends to x_c as $\nu \to 0$. This can easily be verified from the definitions of x_c, $\Delta_{01}(x)$ and x_*.

15. Conclusion

The main physical conclusions of this paper have appeared in the previous two sections. All these conclusions are based on the knowledge of the distribution of all the roots of the equation $\Phi(\zeta, x) = 0$ in the complex ω-plane, which could be obtained only in the interval (x_{**}, x_*) of the values of x, containing the transitional value x_c. It must be pointed out that though x_c has a physical significance as a transitional and a stationary state, neither x_{**} nor x_* has any physical significance. They are only the products of the particular mathematical approach adopted in this paper. Whether one could find a better mathematical approach by which the distribution of the roots of the equation $\Phi(\zeta, x) = 0$ could be known in an interval of values of x more extensive than (x_{**}, x_*), is still open.

It is hoped that the foregoing study will help towards a better understanding of the combined effects of surface-tension, viscosity, and a volume current, as are present in the systems experimentally studied by Lehnert and Sjogren [8], though these systems were in the form of a cylindrical shell. A model much more resembling the experimental set-up has been considered by Murty [9]. However, since the dispersion relations for such cylindrical systems are too complicated to be treated analytically, we have considered in the present paper a system which, apart from gravity, is the form taken by the experimental system as the inner and outer radii tend to infinity keeping a constant difference.

Our condition regarding instability is same as that in the case of a similar inviscid system considered by Murty [1] and that in the case of a similar but compressible and infinitely conducting system with surface-currents considered by Kruskal and Schwarzschild [10].

Appendix A

We have seen in Section 9 that 0, ± 1 are the roots of $\Phi(\zeta, x) = 0$ for every value of x. Apparently this means that $-\nu k^2$ and 0 are possible values of ω for every value of x. This is physically unplausible. Mathematically, too, the values $-\nu k^2$ and 0 are inconsistent with the assumptions made while obtaining the inhomogeneous part of the velocity field from which is derived the dispersion relation (7.1). We shall now show that 0 can be one of the values of ω only when $x = x_c$ and $-\nu k^2$ can be one of the values only at wave numbers where $\Delta_{01}(x) = 0$.

To find the values of k for which $\omega = 0$ can possibly be one of the values of ω, we start with the time-independent perturbation of the type $(a/d) \exp [i(k_x x + k_y y)]$ and proceed to find the condition for consistency of such a perturbation. The steps are similar to those in deriving the dispersion relation except for the fact that the inhomogeneous parts of the solutions of x and y components of velocity are now of the form $[(d/dz)^2 - k^2]^{-1} {\cosh \atop \sinh} kz$, instead of being of the form $[(d/dz)^2 - \lambda^2]^{-1} {\cosh \atop \sinh} kz$. The condition of consistency finally obtained (which can no more be called as a dispersion relation) is the following equation in k:

$$Z(S_{kd} C_{kd} C_{2kd} - kd) = 0. \tag{A.1}$$

The solution $k = 0$ of Eq. (A.1) is trivial. For $k \neq 0$, the second factor in Eq. (A.1) is always positive, and hence x_c (where $Z = 0$) is the only value of x for which 0 is a possible value of ω.

Hence the permanent roots (which correspond to $\omega = 0$) $\zeta = \pm 1$ for all x *need not* be interpreted as giving stationary states *for all* x; but the presence of superposed (second order) roots at $\zeta = \pm 1$ for $x = x_c$ *should be* interpreted as giving a stationary state *at* $x = x_c$.

Similarly if we seek for those k for which $-\nu k^2$ is one of the values of ω, i.e., if we start with a perturbation of the type

$$\frac{a}{d} \exp [- \nu k^2 t + i(k_x x + k_y y)],$$

we obtain the following condition of consistency

$$\Delta_{01}(x) = 0. \tag{A.2}$$

Here the inhomogeneous part of the velocity field has the form $[d^2/dz^2]^{-1} \genfrac{}{}{0pt}{}{\cosh}{\sinh} (kz)$ instead of $[(d/dz)^2 - \lambda^2]^{-1} \genfrac{}{}{0pt}{}{\cosh}{\sinh} kz$.

Equation (A.2) shows that $-\nu k^2$ is a possible value of ω for no other k, than the roots of the equation $\Delta_{01}(2kd) = 0$.

Hence, as mentioned before, the permanent root $\zeta = 0$ for all k need not be interpreted as giving $-\nu k^2$ as one of the values of ω for all k; but the presence of superposed root (third order) at $\zeta = 0$ for $x = x_*$ and other roots of $\Delta_{01}(x) = 0$ should be interpreted as giving a root $\omega = -\nu k^2$.

This proves the statements made in Sections 9, 12.1 and 12.2.

Appendix B

The reason for not interpreting physically the permanent roots 0, ± 1 of $\Phi(\zeta, x) = 0$, when they are of first order, and for interpreting them when they are of higher orders becomes more explicit when the problem of stability is solved as an initial value problem, in the method briefly described below.

We, then, also obtain the expressions (8.1) for the actual time-dependence of quantities on which the concept of the dominant root is based.

In this method we start with a specified initial value for the amplitude of the surface perturbation, and a consistent set of initial values for the amplitudes of perturbation of other quantities. We still assume the space dependence of the type $q(t)\,Q(z) \exp [i(k_x x + k_y y)]$ for all the perturbations and still consider only first order terms. However, instead of assuming from beginning all $q(t)$ to be of the type $q_0 \exp(\omega t)$, we leave the functions $q(t)$ to be determined later by the Laplace transform techniques. It is seen that the problem then reduces to the determination of seven functions: $\alpha_1(t)$, $\alpha_2(t)$, $\beta_1(t)$, $\beta_2(t)$, $D_1(t)$, $D_2(t)$. (These six corresponding to coefficients α_1, α_2, β_1, β_2, D_1, D_2 in Sections 5 and 6) and $a(t)$, the amplitude of the surface perturbation. To this end, we obtain, from boundary conditions, seven linear *inhomogeneous* equations for their Laplace transforms $\alpha_1{}^*(\omega)$, $\alpha_2{}^*(\omega)$, $\beta_1{}^*(\omega), \ldots, a^*(\omega)$, where now ω is the '*transformed*' variable corresponidng to 't.' For a given k the functions $\alpha_1{}^*(\omega), \ldots, a^*(\omega)$ are, thus, obtained in the form

$$\frac{\Delta_j(\omega, k)}{\Delta_8(\omega, k)}, \qquad j = 1, 2, \ldots, 7,$$

where Δ_j are the determinants of appropriate matrices of the coefficients in the abovementioned seven equations.

It is found that $\Delta_8(\omega, k)$ is same as $\mathscr{D}(\omega, k)$ of the dispersion relation (7.1). The functions $\alpha_1(t),\ldots, a(t)$ are then given by the inverse Laplace transforms

$$\int_{\omega_0-i\infty}^{\omega_0+i\infty} \frac{\Delta_j(\omega, k)}{\Delta_8(\omega, k)} \exp(\omega t)\, d\omega, \qquad j = 1,\ldots 7, \tag{B.1}$$

where ω_0 is such that the poles of all the integrals of (B.1) in complex ω-plane lie to the left of the line Re $\omega = \omega_0$. That such ω_0 exists is shown in footnote 1. As $\Delta_j(\omega, k)/\mathscr{D}(\omega, k)$ for each k, is an even function of the expression $(1 + \omega/\nu k^2)^{1/2}$, there is no branch point for any of the integrands. Hence the usual method of contour integration on a semicircle (not passing through any of the series of negative real zeros of $\mathscr{D}(\omega, k)$), followed by application of the residue theorem can still be employed; and we obtain for all the functions $\alpha_1(t)$, $\alpha_2(t),\ldots, a(t)$, the following form:

$$\sum_r A_r(k) \exp\cdot(\omega_r t), \tag{B.2}$$

where ω_r are the poles of the function $\Delta_j/\mathscr{D}$ and $A_r(k)$ are their corresponding residues.

Since all Δ_j are found to be regular everywhere in the finite ω-plane, all the integrands of (B.1) have the *same set of poles*, *viz.*, *the zeros of the function* $\mathscr{D}(\omega, k)$.

Hence the set of eigenfrequencies appearing in (B.2) is the *same* for all the seven functions.

Further, all field perturbations are of the type

$$\sum_1^7 \alpha_1(t) f_1(z) \exp[i(k_x x + k_y y)], \tag{B.4}$$

$\sum_1^7$ denoting summation with respect to $\alpha_1(t)$, $\alpha_2(t),\ldots a(t)$. From (B.2) to (B.4) we see that all the field quantities have the same form of time-dependence, *viz.*,

$$q = \sum_r \mathscr{A}_r(k, z) \exp[\omega_r t + i(k_x x + k_y y)], \tag{B.5}$$

[1] The existence of a number ω_0 such that all the roots of $\mathfrak{D}(\omega, k) = 0$ is the ω-plane lie to the left of the line Re $\omega = \omega_0$, can be proved as follows:

Since $\Phi(\zeta, x) = i\eta^5 \operatorname{Cos}(\eta x) \cosh x$, assymptotically on η-axis; it has a regular distribution of one root per each length π/x of η-axis at sufficiently large distance from origin. This fact together with Pontryagin's theorem proves that for each k, there exists a positive integer K_0 such that *at the most* $(4K_0 + 5)$ roots of $\Phi(\zeta) = 0$ are off the imaginary axis. This in terms of $\mathfrak{D}(\omega, k)$ means that for each k, only a finite subset S_1 of the set S of all the roots of $\mathfrak{D}(\omega, k) = 0$, lies outside the portion $(-\infty, -\nu k^2)$ of the real axis in ω-plane. ω_0 is, then, any number greater than the real part of every member of the subset S_1.

stated in (8.1), the set of ω_r for all the quantities being same, *viz.* the set of zeros of $\mathscr{D}(\omega, k)$. The arbitrary choice of the initial fields at $t = 0$ occurs only in the inhomogeneous terms in the seven algebraic linear equations for $\alpha_1^*(\omega), \ldots a^*(\omega)$, and hence affects only $\Delta_j(\omega, k)$. Hence the result (B.4) is always valid *whatever be the choice of initial fields.*

Equation (B.4) enables us to establish a common criterion of stability and a common concept of growth rate for all the fields in the system, in terms of the dominant, eigenfrequency as mentioned in Section 8.

Returning particularly to $a^*(\omega) = \Delta_7/\mathscr{D}$ and expression Δ_7 and $\mathscr{D}$ in terms of $\zeta = (1 + \omega/\nu k^2)^{1/2}$, it is found that $\zeta = 0, \pm 1$ are permanent roots of $\Delta_7 = 0$ as well as $\mathscr{D} = 0$ for all x. Hence 0 and ± 1 are *not* the poles of $\Delta_7/\mathscr{D}$ in the ω-plane whenever, they are *ordinary* roots of $\Phi(\zeta, x) = 0$; and hence there are no terms of the type $\exp(0 \cdot t)$ and $\exp(-\nu k^2 t)$ in (B.4) for every value of x.

However, for those values of x, for which any of these zeros of $\Phi(\zeta)$ are of order higher than one, that root becomes a pole of $\Delta_7/\mathscr{D}$ in ω-plane, and then only, in (B.4) a corresponding term $\exp(-\nu k^2 t)$ (or $\exp(0 \cdot t)$) appears.

This is the reason for the *physical insignificance*, as mentioned in Section 9 and Appendix A, of the permanent roots $0, \pm 1$ of $\Phi(\zeta, x) = 0$ for all x; except when any of them becomes a zero of higher order (as at $x = x_c$ or when x is the root of $\Delta_{01}(x) = 0$).

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